Circles and Parabolas

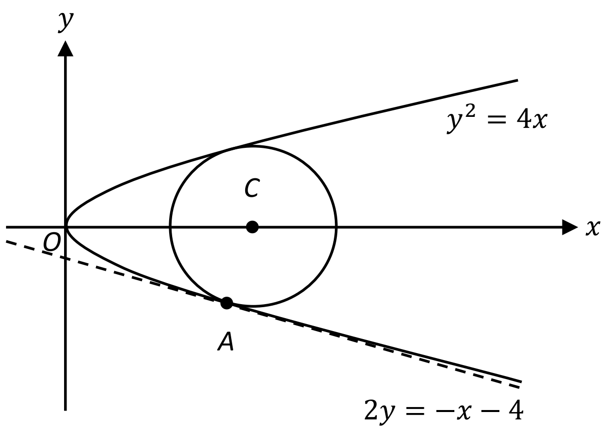
1 Find the possible equations of the circle which passes through the point (4,2) and touches both coordinate axes.

Since the circle touches both axes, the x and y-coordinates of the center of the circle will be equal.
Hence, let the center of the circle be (a, a).
In addition, the radius of the circle will also be a.
Therefore, the equation of the circle is:
$(x-a)^2+(y-a)^2=a^2$
Sub (4,2) into the above equation:
$(4-a)^2+(2-a)^2=a^2$
$16-8a+a^2+4-4a+a^2=a^2$
$a^2-12a+20=0$
$(a-2)(a-10)=0$
$a=2 \text{ or } a=10$
Hence possible equations of circle are:
$(x-2)^2+(y-2)^2=4$ and
$(x-10)^2+(y-10)^2=100$

2 In the diagram (not drawn to scale), the circle with center C touches the curve $y^2 = 4x$ at the point A. $2y = -x - 4$ is the equation of the common tangent to both the circle and the curve at point A. Find

i) the coordinates of point A

ii) the equation of the circle



i) Equate tangent and parabola equations:

$\left(-\frac{x}{2} - 2\right)^2 = 4x$

$\frac{x^2}{4} + 2x + 4 = 4x$

$x^2 - 8x + 16 = 0$

$(x - 4)^2 = 0$

$x = 4$

$y = \pm\sqrt{4(4)} = 4\ (Rej)\ or - 4$

Coordinate of A is $(4, -4)$

ii) Gradient of tangent at A = $-\frac{1}{2}$

Gradient of normal at A = 2

Equation of AC: $y - (-4) = 2(x - 4)$

$y = 2x - 4$

By observation, y-coordinate of C = 0 since it lies on the x-axis.

Sub $y = 0$ into equation AC:

$0 = 2x - 4$

$x = 2$

Coordinate of C = $(2,0)$

Radius of circle = AC = $\sqrt{(4-2)^2 + (-4-0)^2} = \sqrt{20}$

Equation of circle: $(x - 2)^2 + y^2 = 20$

3 The diagram shows circle C_1 (not drawn to scale with the center of the circle at $C(4,-2)$. The line $y = x + 4$ is a tangent to circle at the point A.

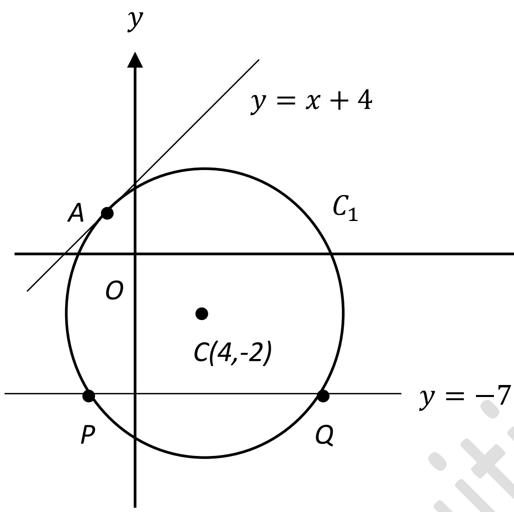
i) Find coordinates of A.

ii) Find the equation of circle C_1.

iii) Given that a line L: $y = -7$ intersects the circle C_1 at the points P and Q, find the coordinates of P and Q and find the shortest distance from the center of the circle to L.

iv) Find the coordinates of the center of a second circle C_2 which has radius $\sqrt{89}$ units and also passes through P and Q. The center of the C_2 is lies above the x-axis.

v) Find $\frac{area\ of\ C_1}{area\ of\ C_2}$



i) Gradient of tangent at A = 1

Gradient of normal at A = $-\frac{1}{1} = -1$

Since normal of tangent passes through the circle center, equation of the normal is:

$(y + 2) = -1(x - 4)$ (tangent ⊥ radius)

$y = -x + 2$

Equate normal and tangent to find A:

$x + 4 = -x + 2$

$x = -1$

$y = -(-1) + 2 = 3$

Point A is $(-1,3)$

ii) Radius of C_1 = $\sqrt{(-1-4)^2 + (3-(-2))^2} = \sqrt{50}\ units$ (Length $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$)

Equation of C_1 = $(x-4)^2 + (y+2)^2 = 50$

iii) Sub $y = -7$ into equation of C_1:

$(x-4)^2 + (-7+2)^2 = 50$

$x^2 + 16 - 8x + 25 = 50$

$x^2 - 8x - 9 = 0$

$x = -1\ or\ x = 9$

Coordinates of P and Q are $(-1,-7)\ and\ (9,-7)$

Shortest distance is to the perpendicular bisector of P and Q.

Mid-point of PQ =$\left(\frac{-1+9}{2}, \frac{-7-7}{2}\right) = (4,-7)$ (Mid-point $= \frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}$)

Distance = $\sqrt{(4-4)^2+(-2-(-7))^2} = 5\ units$

iv) By observation, the x-coordinate of the center of C_2 is the same as the mid-point of P and Q.
Hence, coordinates of Center of C_2 = $(4, y)$

The radius of C_2 = $\sqrt{89} = \sqrt{\left(4-(-1)\right)^2+(y-(-7))^2}$

$89 = 25 + (y+7)^2$

$y + 7 = \pm 8$

$y = 1$ or $y = -8\ (Rej)$

Center of circle C_2 = $(4, 1)$

v) $\frac{area\ of\ C_1}{area\ of\ C_2} = \left(\frac{radius\ of\ C_1}{radius\ of\ c_2}\right)^2 = \left(\frac{\sqrt{50}}{\sqrt{89}}\right)^2$ (for similar Figures, $\frac{Area_1}{Area_2} = \left(\frac{Length_1}{Length_2}\right)^2$)

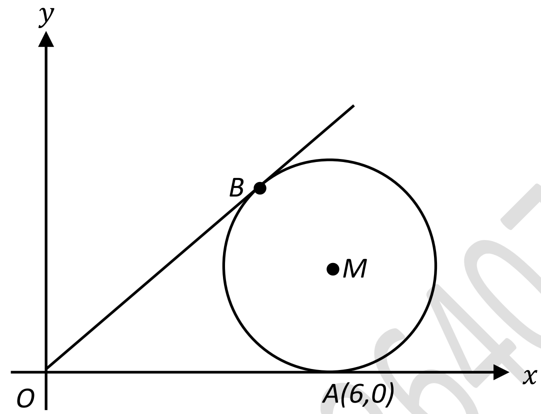
$= \frac{50}{89}$

4 The diagram shows a circle C with center M and radius 2 units. The circle touches the x-axis at the point A(6,0) and the line OB is a tangent to the circle at the point B. Find

i) the coordinates of center M,

ii) the equation of the circle

iii) the equation of the line OB.



i) Center M has to be directly above A, therefore the x-coordinate is 6.
Since the radius of C is 2, the y-coordinate of the center is also 2.
Coordinates of M is (6,2)

ii) Equation of circle: $(x-6)^2 + (y-2)^2 = 4$

iii) Method 1:

$\tan \measuredangle AOM = \frac{2}{6}$

$\measuredangle AOM = 18.435^o$

$\measuredangle AOB = 2 \times \measuredangle AOM = 36.87^o$

Gradient of OB = $\tan \measuredangle AOB = \frac{3}{4}$

Equation of OB : $y = \frac{3}{4}x$

Method 2:

Equation of OB: $y = kx$

Equate OB with circle equation: $(x-6)^2 + (kx-2)^2 = 4$

$x^2 + 36 - 12x + k^2x^2 + 4 - 4kx = 4$

$(1+k^2)x^2 + (-12-4k)x + 36 = 0$

Since OB is tangent to circle C, discriminant = 0

$b^2 - 4ac = 0$

$(-12-4k)^2 - 4(1+k^2)(36) = 0$

$144 + 16k^2 + 96k - 144 - 144k^2 = 0$

$-128k^2 + 96k = 0$

$k = 0\ (Rej)$ or $k = \frac{96}{128} = \frac{3}{4}$

Therefore equation of OB: $y = \frac{3}{4}x$

5 Given that the circle passes through the points $A(1,6)$ and $B(5,8)$ and has radius 5, find the equation of the circle.

Gradient of $AB = \frac{8-6}{5-1} = \frac{1}{2}$
Midpoint of $AB = \left(\frac{1+5}{2}, \frac{6+8}{2}\right) = (3,7)$
Gradient of perpendicular bisector of $AB = -2$
Equation of perpendicular bisector of AB: $y - 7 = -2(x - 3)$
$y = -2x + 13$ -(1)

Let (x, y) be the center of the circle,
$\sqrt{(x-1)^2 + (y-6)^2} = 5$
$(x-1)^2 + (y-6)^2 = 5^2$ -(2)

Sub (1) into (2):
$(x-1)^2 + (-2x + 13 - 6)^2 = 5^2$
$(x-1)^2 + (-2x + 7)^2 = 5^2$
$x^2 + 1 - 2x + 4x^2 + 49 - 28x - 25 = 0$
$5x^2 - 30x + 25 = 0$
$(x-5)(x-1) = 0$

$x = 5, y = 3$
$x = 1, y = 11$

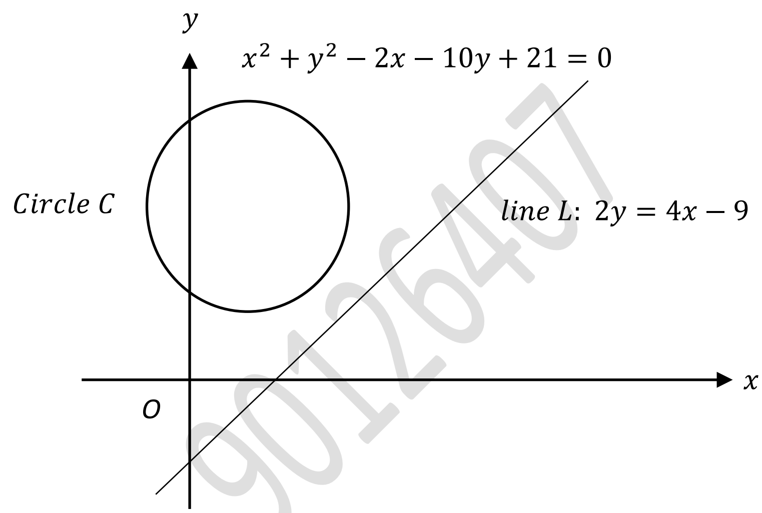
When center is at (5,3),
Equation of circle: $(x-5)^2 + (y-3)^2 = 25$
When center is at (1,11),
Equation of circle: $(x-1)^2 + (y-11)^2 = 25$

6 The diagram shows the circle C with equation $x^2 + y^2 - 2x - 10y + 21 = 0$ and the line L with equation $2y = 4x - 9$.

i) Find the coordinates of the centre and the radius of the Circle C

ii) Find the coordinates of the point on the line L which is closest to the Circle C

iii) Find the equation of the circle which is a reflection of Circle C on the line L.



i) Center of circle $= \left(\frac{-2}{-2}, \frac{-10}{-2}\right) = (1,5)$

Radius of circle $= \sqrt{1^2 + 5^2 - 21} = \sqrt{5}$

ii) Gradient of line L = 2

Gradient of line perpendicular to L $= -\frac{1}{2}$

Equation of line perpendicular to L and passing through center of circle :

$(y - 5) = -\frac{1}{2}(x - 1)$

$y = -\frac{1}{2}x + 5\frac{1}{2}$

$2y = -x + 11$ -(1)

Equate (1) with Line L: $-x + 11 = 4x - 9$

$20 = 5x$

$x = 4$

$y = 3.5$

$(4, 3.5)$ is the point on L which is closest to Circle C

iii) $(4, 3.5)$ is the midpoint between center of Circle C and center of reflected Circle.

$\therefore \left(\frac{1+x}{2}, \frac{5+y}{2}\right) = (4, 3.5)$

$\frac{1+x}{2} = 4$

$x = 7$

$\frac{5+y}{2} = 3.5$

$y = 2$

Center of reflected circle is $(7,2)$

Radius of reflected circle = radius of Circle C = $\sqrt{5}$

Equation of reflected circle: $(x - 7)^2 + (y - 2)^2 = 5$

7 Find the equation of the circle that has its center at $(4,3)$ and has a tangent whose equation is given by $y = 3x + 1$

$y = 3x + 1$ -(1)
Gradient of tangent $= 3$
Gradient of line perpendicular to tangent $= -\frac{1}{3}$
Equation of line passing through center of circle with gradient $-\frac{1}{3}$:
$(y - 3) = -\frac{1}{3}(x - 4)$
$y = -\frac{1}{3}x + 4\frac{1}{3}$ -(2)
Solve (1) and (2) simultaneously:
$x = 1, y = 4$

Radius of circle $= \sqrt{(3-4)^2 + (4-1)^2} = \sqrt{10}$

Equation of circle:
$(x - 4)^2 + (y - 3)^2 = 10$

8 Find the equation of the circle which passes through the origin and the points $A(3,1)$ and $B(-1,-1)$.

Midpoint of $A0 = \left(\frac{3+0}{2}, \frac{1+0}{2}\right) = \left(\frac{3}{2}, \frac{1}{2}\right)$
Gradient of $AO = \frac{1-0}{3-0} = \frac{1}{3}$
Gradient of perpendicular bisector of $AO = -3$
Equation of perpendicular bisector of AO:
$\left(y - \frac{1}{2}\right) = -3\left(x - \frac{3}{2}\right)$
$y = -3x + 5$ -(1)

Midpoint of $B0 = \left(\frac{-1+0}{2}, \frac{-1+0}{2}\right) = \left(\frac{-1}{2}, \frac{-1}{2}\right)$
Gradient of $BO = \frac{-1-0}{-1-0} = 1$
Gradient of perpendicular bisector of $BO = -1$
Equation of perpendicular bisector of BO:
$\left(y - \left(-\frac{1}{2}\right)\right) = -1\left(x - \left(-\frac{1}{2}\right)\right)$
$y = -x - 1$ -(2)

Simultaneously solve (1) and (2):
$x = 3, y = -4$
Center of circle is at $(3, -4)$
Radius of circle = $\sqrt{(1-(-4))^2 - ((3-3)^2} = 5$
Equation of circle: $(y+4)^2 + (x-3)^2 = 25$

9 Circle C has the equation $x^2 + y^2 + 6x - 6y - 7 = 0$.
i) Find the coordinates of the center and the radius of the circle.
ii) Point $A\ (-6, -1)$ is on the circle. The tangent of the circle at point A intersects the y −axis at Point B. Find the coordinates of Point B.
iii) Show that circle C does not intersect the line $y = x - 3$.
iv) Show that the origin lies inside circle C.
v) Find the equation of the circle which is a reflection of circle C on the y −axis.

i) Center of circle $= \left(\frac{6}{-2}, -\frac{6}{-2}\right)$
$= (-3, 3)$
Radius of circle $= \sqrt{(-3)^2 + 3^2 + 7}$
$= \sqrt{25}$
$= 5$
ii) Gradient of normal at Point $A = \frac{3-(-1)}{-3-(-6)}$
$= \frac{4}{3}$
Gradient of tangent at Point $A = -\frac{3}{4}$
Equation of tangent at Point A: $\left(y - (-1)\right) = -\frac{3}{4}(x - (-6))$
$y = -\frac{3}{4}x - \frac{11}{2}$
Point B is the y-intercept, therefore Point B is $\left(0, -\frac{11}{2}\right)$.
iii) Equate $y = x - 3$ with $x^2 + y^2 + 6x - 6y - 7 = 0$:
$x^2 + (x - 3)^2 + 6x - 6(x - 3) - 7 = 0$
$x^2 + x^2 - 6x + 9 + 6x - 6x + 18 - 7 = 0$
$2x^2 - 6x + 20 = 0$
$x^2 - 3x + 10 = 0$
$b^2 - 4ac = (-3)^2 - 4(1)(10) = -31$
Since $b^2 - 4ac < 0$, the line does not intersect circle C.
iv) Length of circle center to origin $= \sqrt{(0 + 3)^2 + (0 - 3)^2}$
$= \sqrt{18} \approx 4.24$

Since $4.24 < 5(Radius)$, the origin lies inside Circle C.
v) Upon reflecting on the y −axis, the center of circle is (3,3)
Therefore the equation of the reflected circle is $(x - 3)^2 + (y - 3)^2 = 25$

10 A, B and C are points with coordinates $(0,-1)$, $(3,-5)$ and $(10,-6)$ respectively. AD is a diameter of a circle with centre at B.
i) Find the equation of the circle.
ii) Show that CD is a tangent to the circle.
iii) E is a point (4,2). Show that CE is another tangent to the circle and state the coordinates of the point of contact.

i) Radius of circle $= \sqrt{(3-0)^2 + \left(-5-(-1)\right)^2}$
$= 5$
Equation of circle: $(x-3)^2 + (\mathrm{y}+5)^2 = 25$ -(1)

ii) B is the midpoint of AD.
$\left(\frac{0+x}{2}, \frac{-1+y}{2}\right) = (3,-5)$
$x = 6$,
$y = -9$
Coordinates of D is $(6,-9)$
Gradient of $CD = \frac{-6-(-9)}{10-6}$
$= \frac{3}{4}$
Gradient of $BD = \frac{-5-(-9)}{3-6}$
$= -\frac{4}{3}$
Since $grad\ of\ CD = \frac{-1}{grad\ of\ BD}$, and D is a common point, CD is tangent to the circle at the point D.
iii) Gradient of $CE = \frac{-6-2}{10-4} = -\frac{4}{3}$
Equation of CE: $(y-2) = -\frac{4}{3}(x-4)$
$y = -\frac{4}{3}x + \frac{22}{3}$ -(2)
Sub (2) into (1):
$(x-3)^2 + \left(-\frac{4}{3}x + \frac{22}{3} + 5\right)^2 = 25$
$x^2 + 9 - 6x + \frac{16}{9}x^2 + \frac{1369}{9} - \frac{296}{9}x - 25 = 0$
$9x^2 + 81 - 54x + 16x^2 + 1369 - 296x - 225 = 0$
$25x^2 - 350x + 1225 = 0$
$x^2 - 14x + 49 = 0$
$(x-7)^2 = 0$
Since the circle and CE intersects at only 1 point (repeated roots0, CE is tangent to the circle.
When $x = 7$, $y = -2$
Hence, the point of contact is $(7,-2)$