Integration

1 Integrate the following:

a) $\int \frac{2x}{2-3x^2} dx$

b) $\int \frac{3-x}{1-x} dx$

a) $\int \frac{2x}{2-3x^2} dx = -\frac{1}{3} \int \frac{-6x}{2-3x^2} dx$

$= -\frac{1}{3} \ln(2-3x^2) + c$

b) $\int \frac{4-x}{2-x} dx = \int \frac{2+2-x}{2-x} dx$

$= \int \frac{2}{2-x} + 1\, dx$

$= -2\ln(2-x) + x + c$

2 a) Show that $\frac{d}{dx} \ln\left(\frac{x^2}{\sin x}\right) = \frac{2}{x} - \cot x$.

b) Hence, find $\int \frac{1}{2} \cot x\, dx$.

a) $\frac{d}{dx} \ln\left(\frac{x^2}{\sin x}\right) = \frac{2x\sin x - x^2 \cos x}{\sin^2 x} \times \frac{\sin x}{x^2}$

$= \frac{2\sin x - x\cos x}{x \sin x}$

$= \frac{2}{x} - \cot x$ (Shown)

b) $\int \frac{1}{2} \cot x\, dx = \frac{1}{2}\left[\int \cot x\, dx + \int \frac{2}{x} dx - \int \frac{2}{x} dx\right]$

$= -\frac{1}{2}\left[\int -\cot x\, dx - \int \frac{2}{x} dx + \int \frac{2}{x} dx\right]$

$= -\frac{1}{2}\left[\int \frac{2}{x} - \cot x\, dx - \int \frac{2}{x} dx\right]$

$= -\frac{1}{2}\left[\ln\left(\frac{x^2}{\sin x}\right) - 2\ln x\right] + c$

$= -\frac{1}{2}\left[\ln \frac{1}{\sin x}\right] + c$

$= \frac{1}{2} \ln \sin x + c$

3 a) Express $\frac{x^3-x^2+3x+9}{x(x^2+3)}$ in partial fractions.

b) Differentiate $\ln(x^2+3)$ with respect to x.

c) Use the results from parts (a) and (b) to find $\int_1^2 \frac{x^3-x^2+3x+9}{x(x^2+3)}\,dx$.

a) $\frac{x^3-x^2+3x+9}{x(x^2+3)} = 1 + \frac{-x^2+9}{x(x^2+3)}$ (by Long division)

$= 1 + \frac{A}{x} + \frac{Bx+C}{x^2+3}$

$-x^2 + 9 = A(x^2+3) + x(Bx+C)$

Sub $x = 0$,

$9 = 3A$

$A = 3$

Comparing x^2-term,

$-x^2 = Ax^2 + Bx^2$

$-1 = 3 + B$

$B = -4$

Sub $x = 1$,

$-1 + 9 = 4A + B + C$

$C = -1 + 9 - 4(3) - (-4)$

$C = 0$

$\therefore \frac{x^3-x^2+3x+9}{x(x^2+3)} = 1 + \frac{3}{x} - \frac{4x}{x^2+3}$

b) $\frac{d}{dx}(\ln(x^2+3)) = \frac{2x}{x^2+3}$

c) $\int_1^2 \frac{x^3-x^2+3x+9}{x(x^2+3)}\,dx = \int_1^2 1 + \frac{3}{x} - \frac{4x}{x^2+3}\,dx$

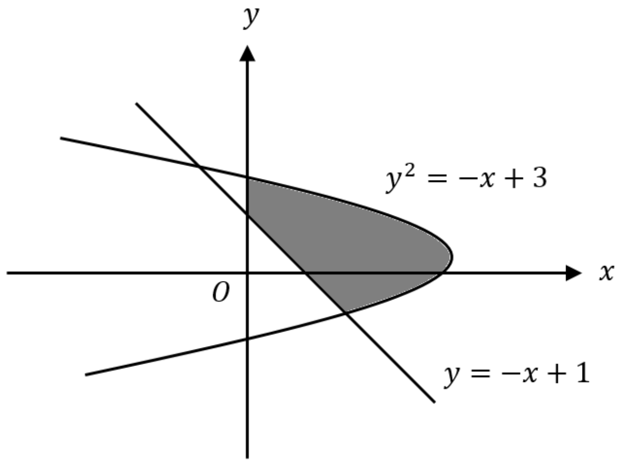
$= \int_1^2 1 + \frac{3}{x}\,dx - 2\int_1^2 \frac{2x}{x^2+3}\,dx$

$= [x + 3\ln x]_1^2 - 2[\ln(x^2+3)]_1^2$

$= (3.079) - (1.119)$

$= 1.96$

4 The diagram shows the curve $y^2 = -x + 3$ and the line $y = -x + 1$. Find the area of the shaded region.



i) $y^2 = -x + 3$ ---(1)
$x = 3 - y^2$
$y = -x + 1$ ---(2)
$x = 1 - y$

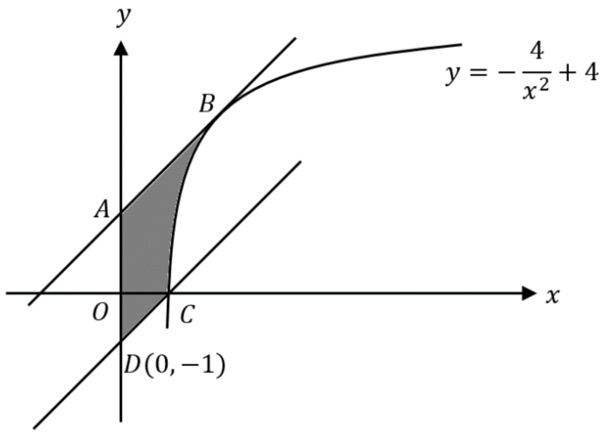
Equate (1) with (2):
$(-x + 1)^2 = -x + 3$
$x^2 + 1 - 2x + x - 3 = 0$
$(x - 2)(x + 1) = 0$
$x = -1$ or $x = 2$
$y = 2$ or $y = -1$
Coordinates of y-intercept of $y = -x + 1$ is (0,1)
To find y-intercepts of $y^2 = -x + 3$:
$y^2 = 0 + 3$
$y = \sqrt{3}$ or $-\sqrt{3}$
Area of Shaded Region $= \int_{-1}^{\sqrt{3}} (3 - y^2) - (1 - y)\, dy - (-\int_{1}^{\sqrt{3}} 1 - y\, dy)$
$= \int_{-1}^{\sqrt{3}} 2 - y^2 + y\, dy + \int_{1}^{\sqrt{3}} 1 - y\, dy$
$= \left[2y - \frac{y^3}{3} + \frac{y^2}{2}\right]_{-1}^{\sqrt{3}} + \left[y - \frac{y^2}{2}\right]_{1}^{\sqrt{3}}$
$= \left(3.232 - \left(-\frac{7}{6}\right)\right) + (0.232 - 0.5)$
$= 4.13\, units^2$

5 The diagram shows the curve $y = -\frac{4}{x^2} + 4$. AB is the tangent to the curve and is parallel to CD. D is the point (0,-1).

i) Find the gradient of CD

ii) Find the coordinates B and A

iii) Find the area of the shaded region ABCD.



i) Sub $y = 0$ into $y = -\frac{4}{x^2} + 4$

$0 = -\frac{4}{x^2} + 4$

$4 = 4x^2$

$x = 1$ or $x = -1$ (Rej)

Gradient of CD $= \frac{0-(-1)}{1-0} = 1$

ii) $y = -\frac{4}{x^2} + 4$

$\frac{dy}{dx} = -\frac{4(-2)}{x^3} = \frac{8}{x^3}$

Since Gradient of AB = Gradient of CD = 1,

$1 = \frac{8}{x^3}$

$x = 2$

$y = 3$

Coordinates of B is $(2,3)$.

Equation of AB: $(y - 3) = 1(x - 2)$

$y = x + 1$

Coordinates of A is $(0,1)$.

iii) Equation of CD: $\big(y - (-1)\big) = 1(x - 0)$

$y = x - 1$

Coordinates of C is $(1,0)$

Area of Shaded Region $= \int_1^2 (x+1) - (-\frac{4}{x^2} + 4)\, dx + \int_0^1 (x+1) - (x-1)\, dx$

$= \int_1^2 x + \frac{4}{x^2} - 3\, dx + \int_0^1 2\, dx$

$= \left[\frac{x^2}{2} - \frac{4}{x} - 3x\right]_1^2 + [2x]_0^1$

$= (-6 - (-6.5)) + (2 - 0)$

$= 2.5\ units^2$

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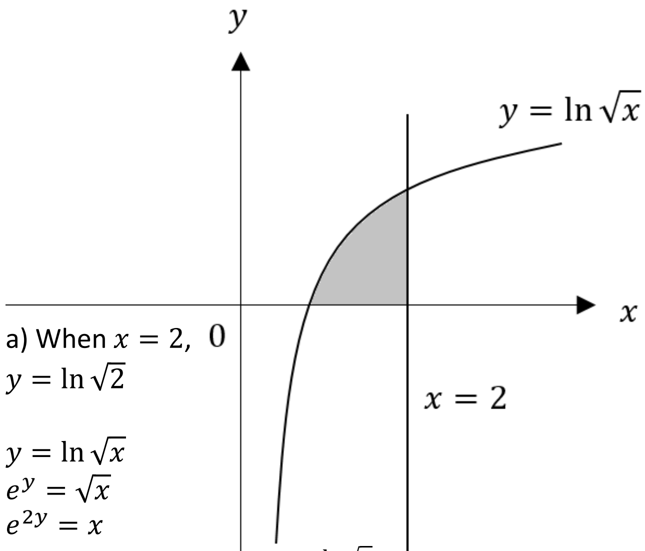
6 a) The shaded region below is bounded by the curve $y = \ln\sqrt{x}$, the x-axis and the line $x = 2$. Show that the area of the shaded region below is, $A = \ln 2 - \frac{1}{2}$.



a) When $x = 2$,

$y = \ln\sqrt{2}$

$y = \ln\sqrt{x}$

$e^y = \sqrt{x}$

$e^{2y} = x$

Area of shaded region $= \int_0^{\ln\sqrt{2}} 2 - e^{2y}\, dy$

$= \left[2y - \frac{e^{2y}}{2}\right]_0^{\ln\sqrt{2}}$

$= 2\ln\sqrt{2} - \frac{e^{2\ln\sqrt{2}}}{2} - 2(0) + \frac{e^{2(0)}}{2}$

$= \ln\sqrt{2}^2 - \frac{e^{\ln 2}}{2} - 0 + \frac{1}{2}$

$= \ln 2 - \frac{2}{2} + \frac{1}{2}$

$= \ln 2 - \frac{1}{2}$

7 The gradient function of a curve is given by $3x^2 + 3x - 18$. The cure has a maximum values of K and a minimum value of -12. Find the value of K.

At turning points, $\frac{dy}{dx} = 0$
$3x^2 + 3x - 18 = 0$
$x^2 + x - 6 = 0$
$(x + 3)(x - 2) = 0$
The 2 turning points are at $x = -3$ and $x = 2$
$\frac{d^2y}{dx^2} = 6x + 3$
When $x = 2, \frac{d^2y}{dx^2} = 15 > 0$
$\therefore$ $(2, -12)$ is a minimum point.
When $x = -3, \frac{d^2y}{dx^2} = -15 < 0$
$\therefore$ $(-3, m)$ is a maximum point.
$y = \int 3x^2 + 3x - 18\,dx$
$= x^3 + \frac{3}{2}x^2 - 18x + c$
Sub $x = 2, y = -12$:
$-12 = 2^3 + \frac{3}{2}(2)^2 - 18(2) + c$
$c = 10$
$\therefore equation\ of\ curve\ is\ y = x^3 + \frac{3}{2}x^2 - 18x + 10$
When $x = -3$,
$y = (-3)^3 + \frac{3}{2}(-3)^2 - 18(-3) + 10$
$y = 50.5$

8 A curve has a turning point at $(2, -10)$ and $\frac{d^2y}{dx^2} = 12x - 6$. Find the equation of the curve.

$\frac{d^2y}{dx^2} = 12x - 6$
$\frac{dy}{dx} = \int 12x - 6\,dx$
$\frac{dy}{dx} = 6x^2 - 6x + c$

Since turning point at $(2, -10)$, $\frac{dy}{dx} = 0$ when $x = 2$:
$0 = 6(2)^2 - 6(2) + c$
$c = -12$
$\therefore \frac{dy}{dx} = 6x^2 - 6x - 12$

$y = \int 6x^2 - 6x - 12\,dx$
$y = 2x^3 - 3x^2 - 12x + c$
Sub $(2, -10)$ into equation:
$-10 = 2(2)^3 - 3(2)^2 - 12(2) + c$
$c = 10$
$\therefore$ equation is $y = 2x^3 - 3x^2 - 12x + 10$

9 The gradient of a function of a curve is given by $\frac{40}{(x-3)^3} - 2$. Given that the line $7y = x + 20$ is a normal to the curve, find the equation of the curve.

$7y = x + 20$
$y = \frac{1}{7}x + \frac{20}{7}$
Gradient of normal $= \frac{1}{7}$
Gradient of tangent $= -7$

$\frac{40}{(x-3)^3} - 2 = -7$
$(x-3)^3 = -8$
$x - 3 = -2$
$x = 1$
Sub $x = 1$ into equation of normal:
$y = \frac{1}{7}(1) + \frac{20}{7}$
$y = 3$
∴ Point $(1,3)$ lies on the curve
$y = \int \frac{40}{(x-3)^3} - 2\,dx$
$y = -\frac{20}{(x-3)^2} - 2x + c$
$3 = -\frac{20}{(1-3)^2} - 2(1) + c$
$c = 10$
∴ Equation of curve is $y = -\frac{20}{(x-3)^2} - 2x + 10$.

10 Solve the following indefinite integrals:

$$\int \sin^2 2x + \frac{1}{\cos^2(x+\frac{\pi}{6})}\,dx$$

$$\int \tan^2\left(x+\frac{\pi}{3}\right) + \cos x \sin x\,dx$$

i) $\int \sin^2 2x + \frac{1}{\cos^2(x+\pi)}\,dx = \int \frac{1-\cos 4x}{2} + \sec^2(x+\frac{\pi}{6})\,dx$

$= \int \frac{1}{2} - \frac{1}{2}\cos 4x + \sec^2(x+\frac{\pi}{6})\,dx$

$= \frac{1}{2}x - \frac{1}{8}\sin 4x + \tan\left(x+\frac{\pi}{6}\right) + c$

ii) $\int \tan^2\left(x+\frac{\pi}{3}\right) + \cos x \sin x\,dx = \int \sec^2\left(x+\frac{\pi}{3}\right) - 1 + \frac{1}{2}\sin(2x)\,dx$

$= \tan\left(x+\frac{\pi}{3}\right) - x - \frac{1}{4}\cos(2x) + c$